short. Finally, the same result can be established on the basis of somewhat weaker but more complicated hypotheses, so the theorem as stated is not the most general one known, and the given conditions are sufficient, but not necessary, for the conclusion to hold.

If each of the functions $F_1, \ldots, F_n$ in Eqs. (11) is a linear function of the dependent variables $x_1, \ldots, x_n$, then the system of equations is said to be **linear**; otherwise, it is **nonlinear**. Thus the most general system of $n$ first order linear equations has the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{14}$$

If each of the functions $g_1(t), \ldots, g_n(t)$ is zero for all $t$ in the interval $I$, then the system (14) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**. Observe that the systems (1) and (2) are both linear. The system (1) is nonhomogeneous unless $F_1(t) = F_2(t) = 0$, while the system (2) is homogeneous. For the linear system (14), the existence and uniqueness theorem is simpler and also has a stronger conclusion. It is analogous to Theorems 2.4.1 and 3.2.1.

**Theorem 7.1.2** If the functions $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval $I\colon \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (14) that also satisfies the initial conditions (13), where $t_0$ is any point in $I$, and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval $I$.

Note that, in contrast to the situation for a nonlinear system, the existence and uniqueness of the solution of a linear system are guaranteed throughout the interval in which the hypotheses are satisfied. Furthermore, for a linear system the initial values $x_1^0, \ldots, x_n^0$ at $t = t_0$ are completely arbitrary, whereas in the nonlinear case the initial point must lie in the region $R$ defined in Theorem 7.1.1.

The rest of this chapter is devoted to systems of linear first order equations (nonlinear systems are included in the discussion in Chapters 8 and 9). Our presentation makes use of matrix notation and assumes that you have some familiarity with the properties of matrices. The basic facts about matrices are summarized in Sections 7.2 and 7.3, and some more advanced material is reviewed as needed in later sections.

## PROBLEMS

In each of Problems 1 through 4, transform the given equation into a system of first order equations.

1. $u'' + 0.5u' + 2u = 0$
2. $u'' + 0.5u' + 2u = 3\sin t$
3. $t^2u'' + tu' + (t^2 - 0.25)u = 0$
4. $u^{(4)} - u = 0$

In each of Problems 5 and 6, transform the given initial value problem into an initial value problem for two first order equations.

5. $u'' + 0.25u' + 4u = 2\cos 3t, \qquad u(0) = 1, \quad u'(0) = -2$
6. $u'' + p(t)u' + q(t)u = g(t), \qquad u(0) = u_0, \quad u'(0) = u'_0$
7. Systems of first order equations can sometimes be transformed into a single equation of higher order. Consider the system

$$x'_1 = -2x_1 + x_2, \qquad x'_2 = x_1 - 2x_2.$$

(a) Solve the first equation for $x_2$ and substitute into the second equation, thereby obtaining a second order equation for $x_1$. Solve this equation for $x_1$ and then determine $x_2$ also.

(b) Find the solution of the given system that also satisfies the initial conditions $x_1(0) = 2$, $x_2(0) = 3$.

(c) Sketch the curve, for $t \geq 0$, given parametrically by the expressions for $x_1$ and $x_2$ obtained in part (b).

In each of Problems 8 through 12, proceed as in Problem 7.

(a) Transform the given system into a single equation of second order.

(b) Find $x_1$ and $x_2$ that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the $x_1x_2$-plane for $t \geq 0$.

8. $x'_1 = 3x_1 - 2x_2, \qquad x_1(0) = 3$
   $x'_2 = 2x_1 - 2x_2, \qquad x_2(0) = \frac{1}{2}$
9. $x'_1 = 1.25x_1 + 0.75x_2, \qquad x_1(0) = -2$
   $x'_2 = 0.75x_1 + 1.25x_2, \qquad x_2(0) = 1$
10. $x'_1 = x_1 - 2x_2, \qquad x_1(0) = -1$
    $x'_2 = 3x_1 - 4x_2, \qquad x_2(0) = 2$
11. $x'_1 = 2x_2, \qquad x_1(0) = 3$
    $x'_2 = -2x_1, \qquad x_2(0) = 4$
12. $x'_1 = -0.5x_1 + 2x_2, \qquad x_1(0) = -2$
    $x'_2 = -2x_1 - 0.5x_2, \qquad x_2(0) = 2$
13. Transform Eqs. (2) for the parallel circuit into a single second order equation.
14. Show that if $a_{11}$, $a_{12}$, $a_{21}$, and $a_{22}$ are constants with $a_{12}$ and $a_{21}$ not both zero, and if the functions $g_1$ and $g_2$ are differentiable, then the initial value problem

$$x'_1 = a_{11}x_1 + a_{12}x_2 + g_1(t), \qquad x_1(0) = x_1^0$$
$$x'_2 = a_{21}x_1 + a_{22}x_2 + g_2(t), \qquad x_2(0) = x_2^0$$

can be transformed into an initial value problem for a single second order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of $t$?

15. Consider the linear homogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y,$$
$$y' = p_{21}(t)x + p_{22}(t)y.$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants $c_1$ and $c_2$. This is the principle of superposition.

15. Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

In each of Problems 16 through 25, find all eigenvalues and eigenvectors of the given matrix.

16. $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

18. $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

19. $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

20. $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

21. $\begin{pmatrix} -3 & 3/4 \\ -5 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

23. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

24. $\begin{pmatrix} 11/9 & -2/9 & 8/9 \\ -2/9 & 2/9 & 10/9 \\ 8/9 & 10/9 & 5/9 \end{pmatrix}$

25. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

Problems 26 through 30 deal with the problem of solving $\mathbf{Ax} = \mathbf{b}$ when $\det \mathbf{A} = 0$.

26. (a) Suppose that $\mathbf{A}$ is a real-valued $n \times n$ matrix. Show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^T\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
*Hint:* You may find it simpler to consider first the case $n = 2$; then extend the result to an arbitrary value of $n$.
(b) If $\mathbf{A}$ is not necessarily real, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.
(c) If $\mathbf{A}$ is Hermitian, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{Ay})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.

27. Suppose that, for a given matrix $\mathbf{A}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. Show that there is also a nonzero vector $\mathbf{y}$ such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

28. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{Ax} = \mathbf{b}$ has solutions. Show that $(\mathbf{b}, \mathbf{y}) = 0$, where $\mathbf{y}$ is any solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Verify that this statement is true for the set of equations in Example 2.
*Hint:* Use the result of Problem 26(b).

29. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Show that if $\boldsymbol{\xi}$ is a solution of $\mathbf{A}\boldsymbol{\xi} = \mathbf{0}$ and $\alpha$ is any constant, then $\mathbf{x} = \mathbf{x}^{(0)} + \alpha\boldsymbol{\xi}$ is also a solution of $\mathbf{Ax} = \mathbf{b}$.

30. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{y}$ is a solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Show that if $(\mathbf{b}, \mathbf{y}) = 0$ for every such $\mathbf{y}$, then $\mathbf{Ax} = \mathbf{b}$ has solutions. Note that this is the converse of Problem 28; the form of the solution is given by Problem 29.
*Hint:* What does the relation $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ say about the rows of $\mathbf{A}$? Again, it may be helpful to consider the case $n = 2$ first.

form (27), provided that there are $n$ linearly independent eigenvectors, but in general all the solutions are complex-valued.

## PROBLEMS

In each of Problems 1 through 6:

(a) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(b) Draw a direction field and plot a few trajectories of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8:

(a) Find the general solution of the given system of equations.

(b) Draw a direction field and a few of the trajectories. In each of these problems, the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

7. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 9 through 14, find the general solution of the given system of equations.

9. $\mathbf{x}' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \mathbf{x}$

11. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}$

13. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 15 through 18, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

15. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

16. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

17. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

18. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$

19. The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second order Euler equation (Section 5.4). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and $r$ must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Referring to Problem 19, solve the given system of equations in each of Problems 20 through 23. Assume that $t > 0$.

20. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

21. $t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

23. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 24 through 27, the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) Sketch a phase portrait of the system.

(b) Sketch the trajectory passing through the initial point (2, 3).

(c) For the trajectory in part (b), sketch the graphs of $x_1$ versus $t$ and of $x_2$ versus $t$ on the same set of axes.

24. $r_1 = -1,\quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix};\qquad r_2 = -2,\quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

25. $r_1 = 1,\quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix};\qquad r_2 = -2,\quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

26. $r_1 = -1,\quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix};\qquad r_2 = 2,\quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

27. $r_1 = 1,\quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix};\qquad r_2 = 2,\quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

28. Consider a $2 \times 2$ system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.

   (a) Note that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, note that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.

   (b) Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.

   (c) Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of $c_1$ and $c_2$ (say $c_1$) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.

   (d) Modify the argument of part (c) if we assume that $c_2 \neq 0$.

   (e) Carry out a similar argument for the case in which the order $n$ is equal to 3; note that the procedure can be extended to an arbitrary value of $n$.

29. Consider the equation

$$ay'' + by' + cy = 0, \tag{i}$$

   where $a$, $b$, and $c$ are constants with $a \neq 0$. In Chapter 3 it was shown that the general solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{ii}$$

   (a) Transform Eq. (i) into a system of first order equations by letting $x_1 = y, x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

## PROBLEMS

In each of Problems 1 through 6:

(a) Express the general solution of the given system of equations in terms of real-valued functions.

(b) Also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8, express the general solution of the given system of equations in terms of real-valued functions.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10, find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the $x_1x_2$-plane.

(c) For your trajectory in part (b), draw the graphs of $x_1$ versus $t$ and of $x_2$ versus $t$.

(d) For your trajectory in part (b), draw the corresponding graph in three-dimensional $tx_1x_2$-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20, the coefficient matrix contains a parameter $\alpha$. In each of these problems:

(a) Determine the eigenvalues in terms of $\alpha$.

(b) Find the critical value or values of $\alpha$ where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of $\alpha$ slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$

16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

17. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix} \mathbf{x}$

18. $\mathbf{x}' = \begin{pmatrix} 3 & \alpha \\ -6 & -4 \end{pmatrix} \mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} \alpha & 10 \\ -1 & -4 \end{pmatrix} \mathbf{x}$

20. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}$

In each of Problems 21 and 22, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

21. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{x}$

22. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

In each of Problems 23 and 24:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the $x_1x_2$-plane. Also draw the trajectories in the $x_1x_3$- and $x_2x_3$-planes.

(c) For the initial point in part (b), draw the corresponding trajectory in $x_1x_2x_3$-space.

23. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \mathbf{x}$

24. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \mathbf{x}$

25. Consider the electric circuit shown in Figure 7.6.6. Suppose that $R_1 = R_2 = 4\ \Omega$, $C = \frac{1}{2}$ F, and $L = 8$ H.

(a) Show that this circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \qquad \text{(i)}$$

where $I$ is the current through the inductor and $V$ is the voltage drop across the capacitor. *Hint:* See Problem 20 of Section 7.1.

(b) Find the general solution of Eqs. (i) in terms of real-valued functions.

(c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 3$ V.

(d) Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

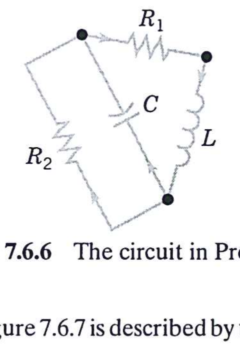


**FIGURE 7.6.6** The circuit in Problem 25.

26. The electric circuit shown in Figure 7.6.7 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}, \qquad \text{(i)}$$

The columns of $\Psi(t)$ are the same as the solutions in Eq. (27) of Section 7.5. Thus the diagonalization procedure does not offer any computational advantage over the method of Section 7.5, since in either case it is necessary to calculate the eigenvalues and eigenvectors of the coefficient matrix in the system of differential equations.

**EXAMPLE 4**

Consider again the system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{45}$$

where $\mathbf{A}$ is given by Eq. (33). Using the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by Eq. (35), you can reduce the system (45) to the diagonal system

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} = \mathbf{D}\mathbf{y}. \tag{46}$$

Obtain a fundamental matrix for the system (46), and then transform it to obtain a fundamental matrix for the original system (45).

By multiplying $\mathbf{D}$ repeatedly with itself, we find that

$$\mathbf{D}^2 = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^3 = \begin{pmatrix} 27 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ldots . \tag{47}$$

Therefore, it follows from Eq. (23) that $\exp(\mathbf{D}t)$ is a diagonal matrix with the entries $e^{3t}$ and $e^{-t}$ on the diagonal; that is,

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}. \tag{48}$$

Finally, we obtain the required fundamental matrix $\Psi(t)$ by multiplying $\mathbf{T}$ and $\exp(\mathbf{D}t)$:

$$\Psi(t) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}. \tag{49}$$

Observe that this fundamental matrix is the same as the one found in Example 1.

## PROBLEMS

In each of Problems 1 through 10:

(a) Find a fundamental matrix for the given system of equations.

(b) Also find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = \mathbf{I}$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$

9. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix} \mathbf{x}$
10. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$
11. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 3.

12. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

by using the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 6.

13. Show that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$, where $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are as defined in this section.
14. The fundamental matrix $\mathbf{\Phi}(t)$ for the system (3) was found in Example 2. Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$ by multiplying $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}(s)$.
15. Let $\mathbf{\Phi}(t)$ denote the fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \mathbf{\Phi}(0) = \mathbf{I}$. In the text we also denoted this matrix by $\exp(\mathbf{A}t)$. In this problem we show that $\mathbf{\Phi}$ does indeed have the principal algebraic properties associated with the exponential function.

(a) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$; that is, show that $\exp(\mathbf{A}t)\exp(\mathbf{A}s) = \exp[\mathbf{A}(t+s)]$. *Hint:* Show that if $s$ is fixed and $t$ is variable, then both $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ and $\mathbf{\Phi}(t+s)$ satisfy the initial value problem $\mathbf{Z}' = \mathbf{A}\mathbf{Z}, \mathbf{Z}(0) = \mathbf{\Phi}(s)$.

(b) Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$; that is, $\exp(\mathbf{A}t)\exp[\mathbf{A}(-t)] = \mathbf{I}$. Then show that $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

(c) Show that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

16. Show that if $\mathbf{A}$ is a diagonal matrix with diagonal elements $a_1, a_2, \dots, a_n$, then $\exp(\mathbf{A}t)$ is also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \dots, \exp(a_n t)$.
17. Consider an oscillator satisfying the initial value problem

$$u'' + \omega^2 u = 0, \qquad u(0) = u_0, \quad u'(0) = v_0. \tag{i}$$

(a) Let $x_1 = u, x_2 = u'$, and transform Eqs. (i) into the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0. \tag{ii}$$

(b) By using the series (23), show that

$$\exp \mathbf{A}t = \mathbf{I}\cos\omega t + \mathbf{A}\frac{\sin\omega t}{\omega}. \tag{iii}$$

(c) Find the solution of the initial value problem (ii).

18. The method of successive approximations (see Section 2.8) can also be applied to systems of equations. For example, consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}^0, \tag{i}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}^0$ is a prescribed vector.

## PROBLEMS

In each of Problems 1 through 4:

(a) Draw a direction field and sketch a few trajectories.

(b) Describe how the solutions behave as $t \to \infty$.

(c) Find the general solution of the system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$

In each of Problems 5 and 6, find the general solution of the given system of equations.

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 7 through 10:

(a) Find the solution of the given initial value problem.

(b) Draw the trajectory of the solution in the $x_1x_2$-plane, and also draw the graph of $x_1$ versus $t$.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

9. $\mathbf{x}' = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the solution of the given initial value problem.

(b) Draw the corresponding trajectory in $x_1x_2x_3$-space, and also draw the graph of $x_1$ versus $t$.

11. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

12. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 13 and 14, solve the given system of equations by the method of Problem 19 of Section 7.5. Assume that $t > 0$.

13. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

14. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}$

15. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 37 in Section 3.4.

16. Consider again the electric circuit in Problem 26 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{L} \\ -\dfrac{1}{C} & -\dfrac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

(a) Show that the eigenvalues are real and equal if $L = 4R^2C$.

(b) Suppose that $R = 1\ \Omega$, $C = 1$ F, and $L = 4$ H. Suppose also that $I(0) = 1$ A and $V(0) = 2$ V. Find $I(t)$ and $V(t)$.

17. Consider again the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x} \qquad \text{(i)}$$

that we discussed in Example 2. We found there that **A** has a double eigenvalue $r_1 = r_2 = 2$ with a single independent eigenvector $\xi^{(1)} = (1, -1)^T$, or any nonzero multiple thereof. Thus one solution of the system (i) is $\mathbf{x}^{(1)}(t) = \xi^{(1)}e^{2t}$ and a second independent solution has the form

$$\mathbf{x}^{(2)}(t) = \xi te^{2t} + \eta e^{2t},$$

where $\xi$ and $\eta$ satisfy

$$(\mathbf{A} - 2\mathbf{I})\xi = \mathbf{0}, \qquad (\mathbf{A} - 2\mathbf{I})\eta = \xi. \qquad \text{(ii)}$$

In the text we solved the first equation for $\xi$ and then the second equation for $\eta$. Here we ask you to proceed in the reverse order.

(a) Show that $\eta$ satisfies $(\mathbf{A} - 2\mathbf{I})^2\eta = \mathbf{0}$.

(b) Show that $(\mathbf{A} - 2\mathbf{I})^2 = \mathbf{0}$. Thus the generalized eigenvector $\eta$ can be chosen arbitrarily, except that it must be independent of $\xi^{(1)}$.

(c) Let $\eta = (0, -1)^T$. Then determine $\xi$ from the second of Eqs. (ii) and observe that $\xi = (1, -1)^T = \xi^{(1)}$. This choice of $\eta$ reproduces the solution found in Example 2.

(d) Let $\eta = (1, 0)^T$ and determine the corresponding eigenvector $\xi$.

(e) Let $\eta = (k_1, k_2)^T$, where $k_1$ and $k_2$ are arbitrary numbers. Then determine $\xi$. How is it related to the eigenvector $\xi^{(1)}$?

**Eigenvalues of Multiplicity 3.** If the matrix **A** has an eigenvalue of algebraic multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \xi e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.